Geometric series of positive linear operators and inverse Voronovskaya theorem

Ulrich Abel, Mircea Ivan and Radu Păltănea

Abstract

We define the associated geometric series for a large class of positive linear operators and study the convergence of the series in the case of sequences of admissible operators. We obtain an inverse Voronovskaya theorem and we apply our results to the Bernstein operators, the Bernstein-Durrmeyer-type operators, and the symmetrical version of Meyer-König and Zeller operators.

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1 Introduction. Basic notions

Let X be a linear subspace of $C[0,1]$ and let $L : X \to X$ be a positive linear operator. Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Denote by L^k, $k \in \mathbb{N}_0$, the iterates of L, defined by $L^0 = I$, where I is the identical operator and $L^k = L \circ \cdots \circ L$, where L appears k times.

While the iterates of L are always well defined, the geometric series of L, namely

$$G_L = \sum_{k=0}^{\infty} L^k \tag{1}$$

needs some restrictions. So, if there is $f \in X$, $f \neq 0$, such that $L(f) = f$, then $G_L(f)$ is not defined.

There exists a rich bibliography concerning the convergence of iterates of positive linear operators, starting with the paper of Kelisky and Rivlin [19] about the iterates of Bernstein operators. A non-exhaustive list of contributions in this direction is given in our references: [2], [5], [8], [9], [10], [11], [13], [14], [15], [21], [22], [27], [28].

Comparatively, the geometric series of positive linear operators have been rather neglected.

In paper Păltănea [23] is made the first study of the convergence of the sequence of geometric series G_{B_n}, $n \in \mathbb{N}$, where B_n, $n \in \mathbb{N}$ are the Bernstein operators. A different approach is given in [4]. A varint of convergence of geometric series of Bernstein operators on a simplex was studied by Raşa [26].

The convergence of geometric series attached to Bernstein-Durrmeyer operators was studied by Abel in [3].

In the present paper we consider geometric series related to a large class of positive linear operators including the majority of the usual approximation operators which preserve linear functions.

Throughout the paper we will use the following notations:

$B[0,1]$, the Banach space of all bounded functions $f\colon [0,1] \to \mathbb{R}$, endowed with the sup norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$;

$C[0,1]$, the Banach space of all continuous functions $f\colon [0,1] \to \mathbb{R}$, endowed with the same sup norm;

$C^k[0,1]$, the space of all functions $f\colon [0,1] \to \mathbb{R}$, possessing a continuous derivative of order k, $k \in \mathbb{N}_0$, where $C^0[0,1] = C[0,1]$;

$C^k(0,1)$, the set of all functions $f : [0,1] \to \mathbb{R}$ which have a continuous k-th derivative on the interval $(0,1)$;

ψ, the function $\psi(x) = x(1-x)$, $x \in [0,1]$;

Π_k, the set of all polynomial functions of degree at most k defined on $[0,1]$;

e_j, the monomials $e_j(t) = t^j$ on $[0,1]$, $j = 0,1,2,\ldots$.

Consider the following linear space of functions:

$$C_\psi[0,1] := \{f : [0,1] \to \mathbb{R} \mid \exists g \in B[0,1] \cap C(0,1) :\ f = \psi g\}.$$

Equivalently, we can write

$$C_\psi[0,1] := \{f \in C[0,1] \mid \exists M > 0 :\ |f(x)| \le M\psi(x),\ x \in [0,1]\}.$$

Note that $\psi C[0,1] \subset C_\psi[0,1]$. On space $C_\psi[0,1]$ we consider the norm

$$\|f\|_\psi := \sup_{x\in(0,1)} \frac{|f(x)|}{\psi(x)},\ f \in C_\psi[0,1].$$

For simplicity, we write sometimes $C_\psi[0,1]$ instead of $(C_\psi[0,1], \|\cdot\|_\psi)$. The following proposition can be immediately proved.

Proposition 1. *The linear space $C_\psi[0,1]$ endowed with the norm $\|\cdot\|_\psi$ is a Banach space.*

Remark 1. However, the linear space $C_\psi[0,1]$ endowed with the norm $\|\cdot\|$ is not a Banach space. For instance the sequence of functions $(\psi f_n)_n$, where f_n are defined by $f_n(x) = n\sqrt{n} \cdot x$, for $x \in [0, \frac{1}{n}]$ and $f_n(x) = \frac{1}{\sqrt{x}}$, for $x \in [\frac{1}{n}, 1]$ is fundamental, but converges uniformly to $\sqrt{x}(1-x)$ which is not an element of $C_\psi[0,1]$.

One of the basic properties of linear positive operators which preserve linear functions is given in the following well-known result.

Theorem A [see, e.g., [23, p. 5]] *If $L : C[0,1] \to C[0,1]$ is a positive linear operator such that $L(e_j) = e_j$, for $j = 0,1$, then we have*

$$L(f)(0) = f(0) \text{ and } L(f)(1) = f(1),\ \text{for all } f \in C[0,1].$$

If X and Y are two subsets of $C[0,1]$, denote $X + Y = \{f + g \mid f \in X, g \in Y\}$. Denote by B_1, the Bernstein operator of order one:

$$B_1(f)(x) = (1-x)f(0) + xf(1),\ f \in C[0,1],\ x \in [0,1]. \tag{2}$$

From the well-known theorem of Korovkin [18, Theorem 8, Chapter 1, §5] we can derive the following immediate consequence.

Theorem B. *If $L : C[0,1] \to C[0,1]$ is a positive linear operator such that $L(\varphi_j) = \varphi_j$, $j = 1, 2, 3$, where $\{\varphi_1, \varphi_2, \varphi_3\}$ is a Chebyshev system on interval $[0,1]$, then $L = I$.*

2 Operators from the class Λ and their iterates

Let $L : C[0,1] \to C[0,1]$ be a positive linear operator preserving constants. By using the Jessen inequality for functionals [17], we obtain

$$0 \le L(\psi) \le \psi. \tag{3}$$

Eq. (3) yields

$$\|L(\psi)\|_\psi \le 1. \tag{4}$$

The previous equations imply that the function

$$b_L(x) = \begin{cases} \frac{L(\psi)(x)}{\psi(x)}, & x \in (0,1), \\ \|L(\psi)\|_\psi, & x \in \{0,1\}, \end{cases}$$

belongs to $B[0,1] \cap C(0,1)$ and

$$L(\psi) = b_L \psi \le \|L(\psi)\|_\psi \psi. \tag{5}$$

Definition 1. *Denote by Λ the class of all positive linear operators $L : C[0,1] \to C[0,1]$ satisfying the following conditions:*

(a) *L preserves linear functions;*

(b) $\|L(\psi)\|_\psi < 1$*;*

(c) $L \ne B_1$.

Lemma 1. *If $L \in \Lambda$, then the following properties are satisfied:*

(i) $\|L(\psi)\|_\psi > 0$*;*

(ii) $L(C_\psi[0,1]) \subset C_\psi[0,1]$*;*

(iii) $\|L\|_{\mathcal{L}(C_\psi[0,1], C_\psi[0,1])} = \|L(\psi)\|_\psi = \|b_L\|$.

Proof.

(i) Suppose, for reductio ad absurdum, that $\|L(\psi)\|_\psi = 0$. It follows that $b_L = 0$, hence $L(\psi) = 0$. We show that this implies $L = B_1$. Indeed, let $f \in C^1[0,1]$ and denote $M = 2\|(f - B_1(f))'\|$. Since $f(0) = B_1(f)(0)$ we obtain for $t \in [0, \frac{1}{2}]$ that $|f(t) - B_1(f)(t)| \le \|(f - B_1(f))'\| t \le M\psi(t)$. The same inequality can be proved for $t \in [\frac{1}{2}, 1]$. Then, for

$t \in [0,1]$, we deduce $|L(f)(t) - B_1(f)(t)| = |L(f - B_1(f))(t)| \leq L(|f - B_1(f)|)(t) \leq ML(\psi)(t) = 0$. Hence $L(f) = B_1(f)$, for $f \in C^1[0,1]$. Since $C^1[0,1]$ is dense in $C[0,1]$ it follows $L = B_1$. But this contradicts the hypothesis. It follows that $\|L(\psi)\|_\psi > 0$.

(*ii*) Let $g \in B[0,1] \cap C(0,1)$. We have $|L(\psi g)| \leq \|g\| L(\psi) \leq \|g\| \, \|L(\psi)\|_\psi \, \psi$. It follows that $L(\psi g) \in C_\psi[0,1]$.

(*iii*) From above, for $g \in B[0,1] \cap C(0,1)$, we obtain:

$$\sup_{x \in (0,1)} \frac{|L(\psi g)(x)|}{\psi(x)} \leq \|L(\psi)\|_\psi \sup_{x \in (0,1)} |g(x)| = \|L(\psi)\|_\psi \, \|\psi g\|_\psi,$$

i.e.,

$$\|L(f)\|_\psi \leq \|L(\psi)\|_\psi \, \|f\|_\psi, \qquad f \in C_\psi[0,1].$$

But $\|L(\psi)\|_\psi = \|b_L\|$. For $f = \psi$, we have equality and the proof is complete. □

Definition 2. *Denote by Λ_0, the class of positive linear operators $L : C[0,1] \to C[0,1]$ which satisfy the following conditions:*

(a_1) *L preserves linear functions;*

(b_1) *$L(\Pi_2) \subset \Pi_2$;*

(c_1) *$L \neq I$ and $L \neq B_1$.*

Lemma 2. *We have $\Lambda_0 \subset \Lambda$. Moreover, Λ_0 coincides with the class of those operators $L \in \Lambda$, for which b_L is a constant function with value in interval $(0,1)$.*

Proof. Let $L \in \Lambda_0$. By (5), we have $L(\psi) = b_L \, \psi$. Since $L\psi \in \Pi_2$, we deduce that there exist constants a, b, c such that $b_L \psi = a + b\, e_1 + c\, \psi$. From Theorem A it follows $b_L(x) = c$, $x \in (0,1)$. Taking into account the definition of b_L, we deduce that $b_L = c\, e_0$.

From relation (3) we obtain $c \leq 1$. If $c = 1$, then operator L preserves functions e_0, e_1, ψ which form a Chebyshev system on interval $[0,1]$ and by Theorem B it follows $L = I$. Contradiction to (c_1). Therefore $\|L(\psi)\|_\psi = c < 1$. If $c = 0$, which is equivalent to condition $\|L(\psi)\|_\psi = 0$, then like in the proof of Lemma 1 we obtain $L = B_1$. Contradiction. Consequenly we have $c \in (0,1)$ and $L \in \Lambda$.

Conversely, if for $L \in \Lambda$, there is a constant $c \in (0,1)$ such that $b_L(x) = c$, for $x \in [0,1]$, it follows that L satisfies condition (b_1), since the functions e_0, e_1, ψ forms a basis of Π_2. Also, since $L(\psi) \neq \psi$ we have $L \neq I$. □

Theorem 1. *If $L \in \Lambda$ we have*

$$\lim_{k \to \infty} \|L^k(f) - B_1(f)\|_\psi = 0, \qquad \text{for all } f \in C_\psi[0,1] + \Pi_1. \tag{6}$$

Proof. Let $f = h + P_1$, where $h \in C_\psi[0,1]$ and $P_1 \in \Pi_1$. Since L^k and B_1 preserve P_1 we have:

$$L^k(f) - B_1(f) = L^k(h) - B_1(h) = L^k(h - B_1(h)).$$

Since $h - B_1(h)$ belongs to $C_\psi[0,1]$, by using Lemma 1 (ii) and Eq. (5) repetitively, we obtain:

$$\|L^k(f) - B_1(f)\|_\psi = \|L^k(h - B_1(h))\|_\psi \leq \left(\|L(\psi)\|_\psi\right)^k \|h - B_1 h\|_\psi.$$

Since $\|L(\psi)\|_\psi < 1$, passing to limit as $k \to \infty$ the proof of (6) is complete. □

We mention a different result for the convergence of iterates of operators given by Gavrea and Ivan [10], which in a slightly modified form could be read as follows.

Theorem C *If $L : C[0,1] \to C[0,1]$ is a positive linear operator such that $L(e_j) = e_j$, $j = 0, 1$, and there is $m > 0$ and $q \geq 1$ such that $\psi - L(\psi) \geq m\,\psi^q$, then we have*

$$\lim_{k\to\infty} \|L^k(f) - B_1(f)\| = 0, \text{ for all } f \in C[0,1]. \tag{7}$$

Note that, for $q = 1$, the conditions in Theorem C are satisfied by operators in class Λ.

However Theorem 1 is not a consequence of Theorem C, since it is given with respect to norm $\|\cdot\|_\psi$.

3 Geometric series of operators from class Λ

We consider the geometric series G_L attached to an operator $L \in \Lambda$ on the space $C_\psi[0,1]$.

Theorem 2. *For any $L \in \Lambda$ we have:*

(i) *the operator $G_L : C_\psi[0,1] \to C_\psi[0,1]$, given in (1) is well defined if we consider the convergence with regard to the norm $\|\cdot\|_\psi$ and hence also with regard to norm $\|\cdot\|$;*

(ii) *the operator G_L is positive and linear;*

(iii) $(1 - \|b_L\|)G_L(\psi) \leq \psi$;

(iv) $\|G_L\|_{\mathcal{L}(C_\psi[0,1], C_\psi[0,1])} \leq (1 - \|b_L\|)^{-1}$.

Proof. (i) Let $f \in C_\psi[0,1]$. From Lemma 1 (ii) it follows by induction that $L^k(f) \in C_\psi[0,1]$, for all $k \geq 0$. Hence the partial sums of the series $\sum_{k=0}^\infty L^k(f)$ are in $C_\psi[0,1]$. From Lemma 1 (iii) we obtain $\|L^k\|_{\mathcal{L}(C_\psi[0,1], C_\psi[0,1])} \leq \|b_L\|^k$, for $k \geq 0$. It follows that

$$\sum_{k=0}^{\infty} \|L^k\|_{\mathcal{L}(C_\psi[0,1], C_\psi[0,1])} \leq \frac{1}{1 - \|b_L\|} \tag{8}$$

(note that $\|b_L\| < 1$ by Definition 1 (b)). Consequently, the series $\sum_{k=0}^\infty L^k(f)$ is convergent in the space $C_\psi[0,1]$, with regard to the norm $\|\cdot\|_\psi$.

(ii) It is immediate.

(iii) It follows from Eq. (5).

(iv) The inequality is a consequence of Eq. (8). □

Theorem 3. *For any operator $L \in \Lambda$, the following equalities are true on the Banach space $(C_\psi[0,1], \|\cdot\|_\psi)$:*

$$\begin{aligned} (I-L)\circ G_L &= I; & (9)\\ G_L \circ (I-L) &= I. & (10)\end{aligned}$$

Proof. Since $\|L^k\|_{\mathcal{L}(C_\psi[0,1],C_\psi[0,1])} \le \|b_L\|^k$, for $k \in \mathbb{N}_0$, we obtain

$$\lim_{k\to\infty} \|L^k(f)\|_\psi = 0, \qquad \text{for all } f \in C_\psi[0,1].$$

Then, using the continuity of operator $I-L$ we obtain successively, for $f \in C_\psi[0,1]$):

$$\begin{aligned} ((I-L)\circ G_L)(f) &= (I-L)\left(\lim_{m\to\infty}\sum_{k=0}^m L^k(f)\right)\\ &= \lim_{m\to\infty}(I-L)\left(\sum_{k=0}^m L^k(f)\right)\\ &= \lim_{m\to\infty}(I-L^{m+1})(f)\\ &= f.\end{aligned}$$

For $f \in C_\psi[0,1]$, we have

$$(G_L\circ(I-L))(f) = \lim_{m\to\infty}\sum_{k=0}^m (L^k\circ(I-L))(f) = \lim_{m\to\infty}(I-L^{m+1})(f) = f. \quad \square$$

4 Convergence of geometric series on the space $C_\psi[0,1]$

For $f \in C(0,1)\cap B[0,1]$ and $x \in [0,1]$ define

$$F(f)(x) = (1-x)\int_0^x tf(t)dt + x\int_x^1 (1-t)f(t)dt. \qquad (11)$$

In [24], the following immediate result is proved.

Lemma E *For any $f \in C[0,1]$, we have $F(f) \in \psi C[0,1]\cap C^2[0,1]$ and*

$$F''(f) = -f. \qquad (12)$$

A slightly modified version of it is the following.

Lemma 3. *For any $f \in B[0,1]\cap C(0,1)$, we have $F(f) \in \psi C[0,1]\cap C^2(0,1)$ and*

$$F''(f)(x) = -f(x), \qquad x \in (0,1). \qquad (13)$$

Proof. Let $f \in B[0,1] \cap C(0,1)$. The function $g := F(f)\psi^{-1}$ defined and continuous on $(0,1)$ can be extended by continuity to the interval $[0,1]$, since $\lim\limits_{x\to 0+} g(x) = \int_0^1 (1-t)f(t)dt$ and $\lim\limits_{x\to 1-} g(x) = \int_0^1 tf(t)dt$. Relation (13) follows immediately. $\square$

Consider a sequence of operators $(L_n)_n$, $L_n \in \Lambda$, $n \in \mathbb{N}$. For simplicity, we denote

$$G_n := G_{L_n},\ \alpha_n := 1 - b_{L_n},\ \nu_n := \min_{x\in[0,1]} \alpha_n, \tag{14}$$

$$\eta_n := \sup_{x,y\in[0,1]} |\alpha_n(x) - \alpha_n(y)|/\nu_n, \tag{15}$$

$$M_n^k(x) := L_n((e_1 - xe_0)^k, x), \tag{16}$$

for $x \in [0,1]$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0$. Then $M_n^2 = \alpha_n \psi$. With these notations we have:

Theorem 4. *Let the sequence of operators $(L_n)_n$, $L_n \in \Lambda$, $n \in \mathbb{N}$, which satisfies the following "little-o" conditions:*

$$M_n^4(x) = \mathrm{o}(M_n^2(x)),\ \textit{uniformly with regard to } x \in [0,1]; \tag{17}$$

$$\eta_n = \mathrm{o}(1), \tag{18}$$

as $n \to \infty$. Then, for any $f \in B[0,1] \cap C(0,1)$, we have

$$\lim_{n\to\infty} \|\alpha_n G_n(\psi f) - 2F(f)\|_\psi = 0. \tag{19}$$

Proof. We split the proof in two steps.

Step 1. We prove relation (19) in the case $f \in C[0,1]$. Set $g = \psi f$. Fix $n \in \mathbb{N}$. Denote for simplicity $f_n := f/\alpha_n$ and $H_n := F(f_n)$. It follows $f_n \in B[0,1] \cap C(0,1)$ and using Lemma 3 we have $H_n \in C^2(0,1) \cap C_\psi[0,1]$. We obtain for $y \in (0,1)$ and $t \in [0,1]$:

$$H_n(t) = H_n(y) + H_n'(y)(t-y) + \frac{1}{2} \cdot H_n''(y)(t-y)^2 + \Theta_{n,y}(t),$$

where

$$\Theta_{n,y}(t) := \int_y^t (t-u)(H_n''(u) - H_n''(y))\, du,$$

and $H_n''(t) = -f_n(t)$, for $t \in (0,1)$. Fix $y \in (0,1)$. Applying the linear functional $L_n(\cdot, y)$, and taking into account condition (a) from Definition 1 we obtain

$$L_n(H_n)(y) = H_n(y) - \frac{1}{2} f_n(y) M_n^2(y) + L_n(\Theta_{n,y})(y).$$

It follows

$$(I - L_n)(H_n)(y) = \frac{1}{2} g(y) - L_n(\Theta_{n,y})(y). \tag{20}$$

Note that Eq. (20) which was derived for $y \in (0,1)$ is also true for $y = 0$ or $y = 1$, since in these cases the both side terms of the equation vanish.

Denote $S_n(y) = L_n(\Theta_{n,y})(y)$, $y \in [0,1]$. Since $H_n \in C_\psi[0,1]$, we obtain $(I - L_n)(H_n) \in C_\psi[0,1]$ and then we deduce $S_n \in C_\psi[0,1]$. Applying operator G_n, to the functions in both sides of Eq. (20) and taking into account Theorem 3 we obtain, for $x \in [0,1]$:

$$H_n(x) = \frac{1}{2}\, G_n(g)(x) - G_n(S_n)(x).$$

Hence

$$\|\alpha_n\big(G_n(g) - 2F(f_n)\big)\|_\psi = 2\|\alpha_n G_n(S_n)\|_\psi. \tag{21}$$

Let $\omega(f,\delta) := \sup\{|f(u) - f(v)|,\ u,v \in [0,1],\ |u-v| \le \delta\}$, for $\delta > 0$, be the modulus of continuity of f. By condition (17), there is a sequence $(\beta_n)_n$ of positive numbers such that $\beta_n \to 0$ $(n \to \infty)$ and $M_n^4(x) \le \beta_n M_n^2(x)$, for all $x \in [0,1]$ and $n \in \mathbb{N}$.

For $u, y \in [0,1]$, we obtain

$$\begin{aligned} |f_n(u) - f_n(y)| &\le \Big| f(u)\Big(\frac{1}{\alpha_n(u)} - \frac{1}{\alpha_n(y)}\Big)\Big| + \frac{|f(u) - f(y)|}{\alpha_n(y)} \\ &\le \frac{\eta_n\|f\|}{\alpha_n(y)} + \frac{1}{\alpha_n(y)}\Big(1 + \frac{(u-y)^2}{\beta_n}\Big)\, \omega(f, \sqrt{\beta_n}). \end{aligned}$$

Consequently, for $t \in [0,1]$, by Lemma 3, we have:

$$\begin{aligned} |\Theta_{n,y}(t)| &\le \Big| \int_y^t (t-u)|f_n(u) - f_n(y)|\, du \Big| \\ &\le \frac{1}{2\alpha_n(y)}\Big[\eta_n\|f\|(t-y)^2 + \Big((t-y)^2 + \frac{(t-y)^4}{6\beta_n}\Big)\omega(f, \sqrt{\beta_n})\Big]. \end{aligned}$$

Then we deduce:

$$\begin{aligned} |S_n(y)| = |L_n(\Theta_{n,y})(y)| &\le \frac{1}{2\alpha_n(y)}\Big[\eta_n\|f\|M_n^2(y) + \Big(M_n^2(y) + \frac{1}{6\beta_n}\cdot M_n^4(y)\Big)\omega(f, \sqrt{\beta_n})\Big] \\ &\le \psi(y)\Big[\frac{1}{2}\eta_n\|f\| + \frac{7}{12}\omega(f, \sqrt{\beta_n})\Big], \end{aligned}$$

because $M_n^2 = \alpha_n\psi$. Therefore, by Th. 2 (iii), it follows that

$$\begin{aligned} \|\alpha_n G_n(S_n)\|_\psi &\le \Big(\frac{1}{2}\eta_n\|f\| + \frac{7}{12}\omega(f, \sqrt{\beta_n})\Big)\, \|\alpha_n G_n(\psi)\|_\psi \\ &\le \Big(\frac{1}{2}\eta_n\|f\| + \frac{7}{12}\omega(f, \sqrt{\beta_n})\Big)\, \frac{\|\alpha_n\|}{\nu_n} \\ &\le \Big(\frac{1}{2}\eta_n\|f\| + \frac{7}{12}\omega(f, \sqrt{\beta_n})\Big)\, (1 + \eta_n). \end{aligned} \tag{22}$$

Since $f \in C[0,1]$ we have $\lim_{\rho \to 0} \omega(f,\rho) = 0$. From relations (17), (18), (21) and (22) it results:

$$\lim_{n\to\infty} \|\alpha_n\big(G_n(g) - 2F(f_n)\big)\|_\psi = 0. \tag{23}$$

Finally, we use the inequality

$$\begin{aligned}\|\alpha_n G_n(g) - 2F(f)\|_\psi &\leq \|\alpha_n\big(G_n(g) - 2F(f_n)\big)\|_\psi \\ &\quad +2\|\alpha_n F(f_n) - F(f)\|_\psi. \end{aligned} \tag{24}$$

For $y \in (0,1)$, we obtain

$$\begin{aligned}|\psi^{-1}(y)\big(\alpha_n(y)F(f_n)(y) - F(f)(y)\big)| &= \Big|\psi^{-1}(y)F\Big(f \cdot \frac{\alpha_n(y)e_0 - \alpha_n}{\alpha_n}\Big)(y)\Big| \\ &\leq \psi^{-1}(y)F\Big(|f| \cdot \frac{|\alpha_n(y)e_0 - \alpha_n|}{\alpha_n}\Big)(y) \\ &\leq \eta_n \psi^{-1}(y)F(|f|)(y) \\ &\leq \eta_n \|F(|f|)\|_\psi.\end{aligned}$$

Hence, by using assumption (18) it follows that

$$\lim_{n\to\infty} \|\alpha_n F(f_n) - F(f)\|_\psi = 0. \tag{25}$$

From Eqs. (24), (23) and (25) we obtain (19).

Step 2. Now we prove relation (19) in the general case when $f \in B[0,1] \cap C(0,1)$. Let ε, $0 < \varepsilon < 1$. Then choose a number δ with $0 < \delta < \frac{\varepsilon}{96(\|f\|+1)}$. Define

$$f_\delta(t) := \begin{cases} f(\delta), & t \in [0,\delta] \\ f(t), & t \in [\delta, 1-\delta] \\ f(1-\delta), & t \in [1-\delta, 1]. \end{cases}, \qquad \varphi_\delta^1(t) := \begin{cases} 1, & t \in [0,\delta] \\ 2 - t/\delta, & t \in [\delta, 2\delta] \\ 0, & t \in [2\delta, 1] \end{cases}.$$

Then define the function $\varphi_\delta^2(t) = \varphi_\delta^1(1-t)$, $t \in [0,1]$. Note that:

$$0 \leq F(\varphi_\delta^1)(x) \leq (1-x)\int_0^{\min\{x,2\delta\}} t\,dt + x\int_{\min\{x,2\delta\}}^{2\delta} (1-t)\,dt \leq (1-x)x\delta + 2x(1-x)\delta.$$

It follows $\|F(\varphi_\delta^1)\|_\psi \leq 3\delta$. In an analogous mode we have $\|F(\varphi_\delta^2)\|_\psi \leq 3\delta$.

Since $f_\delta, \varphi_\delta^1, \varphi_\delta^2 \in C[0,1]$, using Step 1, there is $n_\varepsilon \in \mathbb{N}$, such that, for any integer $n \geq n_\varepsilon$, we have

$$\|\alpha_n G_n(\psi f_\delta) - 2F(f_\delta)\|_\psi < \frac{\varepsilon}{3}, \text{ and } \|\alpha_n G_n(\psi\varphi_\delta^j) - 2F(\varphi_\delta^j)\|_\psi < \frac{\varepsilon}{24(\|f\|+1)}, \quad j = 1,2.$$

Let now $n \geq n_\varepsilon$. We have

$$\begin{aligned}
\|\alpha_n G_n(\psi f) - 2F(f)\|_\psi &\leq \|\alpha_n G_n(\psi f_\delta) - 2F(f_\delta)\|_\psi \\
&\quad + \|\alpha_n G_n(\psi(f - f_\delta))\|_\psi + 2\|F(f - f_\delta)\|_\psi \\
&< \frac{\varepsilon}{3} + \sum_{j=1}^{2} \|\alpha_n G_n(\psi|f - f_\delta|\varphi_\delta^j)\|_\psi + 2\|F(f - f_\delta)\|_\psi \\
&\leq \frac{\varepsilon}{3} + 2\|f\| \sum_{j=1}^{2} \|\alpha_n G_n(\psi\varphi_\delta^j)\|_\psi + 4\|f\| \sum_{j=1}^{2} \|F(\varphi_\delta^j)\|_\psi \\
&\leq \frac{\varepsilon}{3} + 2\|f\| \sum_{j=1}^{2} \left[\|\alpha_n G_n(\psi\varphi_\delta^j) - 2F(\varphi_\delta^j)\|_\psi + 4\|F(\varphi_\delta^j)\|_\psi\right] \\
&\leq \frac{\varepsilon}{3} + 2\|f\| \left[2 \cdot \frac{\varepsilon}{24(\|f\| + 1)} + 8 \cdot 3\delta\right] \\
&< \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily chosen it follows relation (19) in the general case when $f \in B[0,1] \cap C(0,1)$. □

Corollary 1. *In conditions of Theorem 4 there is a constant $M > 0$, such that*

$$\|\alpha_n G_n\|_{\mathcal{L}(C_\psi[0,1], C_\psi[0,1])} \leq M, \qquad \text{for all } n \in \mathbb{N}. \tag{26}$$

Proof. We can apply the uniform boundedness principle to operators $\alpha_n G_n : C_\psi[0,1] \to C_\psi[0,1]$ by taking into account relation (19) and the fact that $C_\psi[0,1]$ is a Banach space. □

Remark 2. By taking into account Remark 1 we cannot derive by the uniform boundedness principle a bound for $\|\alpha_n G_n\|_{\mathcal{L}((C_\psi[0,1], \|\cdot\|), (C_\psi[0,1], \|\cdot\|))}$, for all $n \in \mathbb{N}$.

5 Inverse Voronovskaya theorem

An inverse Voronovskaya theorem was established using the theory of semigroups of operators, for instance in [6]. Here we give a variant of inverse Voronovskaya theorem using the geometric series of operators. We use the notations given in (14) and (15).

The Voronvskaya theorem can be expressed in the strong form of the convergence in Banach space $(C_\psi[0,1], \|\cdot\|_\psi)$. From the general result given in [12, Corollary 4.3], we deduce, with the notations in this paper:

Theorem D. *Let a sequence of positive linear operators $(L_n)_n$, $L_n \in \Lambda_0$, with the additional conditions: $L(C[0,1]) \subset C^4[0,1]$, $L(\Pi_j) \subset \Pi_j$, $j = 3, 4$ and relation (17) is true. Then, for any $f \in C^2[0,1]$, we have:*

$$\lim_{n\to\infty} \left\| \frac{1}{\nu_n}(L_n(f) - f) - \frac{1}{2} f'' \psi \right\|_\psi = 0. \tag{27}$$

In this section we state a converse result to Theorem D.

Theorem 5. *Let $(L_n)_n$ be a sequence of operators $L_n \in \Lambda$, $n \in \mathbb{N}$, which satisfy conditions (17), (18) and also the following condition:*

$$L_n(\psi|\alpha - \alpha(x)e_0|)(x) = \mathrm{o}(\nu_n^2\psi(x)), \text{ uniformly for } x \in [0,1]. \tag{28}$$

If for $f \in C_\psi[0,1] + \Pi_1$ there holds

$$\lim_{n\to\infty} \left\| \frac{1}{\alpha_n}(L_n(f) - f) - g\psi \right\|_\psi = 0, \tag{29}$$

with a certain $g \in C(0,1) \cap B[0,1]$, then $f \in C^2(0,1)$ and $g(x) = \frac{1}{2}f''(x)$, for $x \in (0,1)$.

Proof. Let $f \in C_\psi[0,1] + \Pi_1$ and $g \in C(0,1) \cap B[0,1]$ satisfying the conditions in the theorem. Put $f_1 = f - B_1(f)$ and $h_n = \frac{1}{\alpha_n}(L_n(f) - f)$, for $n \in \mathbb{N}$. We have $f_1 \in C_\psi[0,1]$ and $h_n \in C_\psi[0,1]$. Then, there is $g_1 \in C(0,1) \cup B[0,1]$, such that $f_1 = \psi g_1$. First we prove the following limit

$$\lim_{n\to\infty} \|\alpha_n G_n(h_n) + f_1\|_\psi = 0. \tag{30}$$

Indeed, we can write

$$\alpha_n G_n(h_n) = \alpha_n \sum_{k=0}^{\infty} L_n^k \left(\frac{L_n(f_1) - f_1}{\alpha_n} \right) = -f_1 + \sum_{k=0}^{\infty} \alpha_n L_n^k \left(\frac{1}{\alpha_n} L_n(f_1) - L_n\left(\frac{f_1}{\alpha_n}\right) \right).$$

By condition (28), there is a sequence $(\gamma_n)_n$ of positive numbers converging to zero such that $L_n(\psi|\alpha - \alpha(x)e_0|)(x) \le \gamma_n \nu_n^2 \psi(x)$, for all $x \in [0,1]$ and $n \in \mathbb{N}$.

For $x \in [0,1]$ and $n \in \mathbb{N}$, we have:

$$\begin{aligned} \left| \frac{1}{\alpha_n(x)} \cdot L_n(f_1)(x) - L_n\left(\frac{f_1}{\alpha_n}\right)(x) \right| &\le L_n\left(|f_1| \cdot \frac{|\alpha_n - \alpha_n(x)e_0|}{\alpha_n \alpha_n(x)} \right)(x) \\ &\le \|g_1\| \gamma_n \psi(x). \end{aligned}$$

Then, using relation (iii) of Theorem 2, we have:

$$\begin{aligned} \left| \sum_{k=0}^{\infty} \alpha_n L_n^k \left(\frac{1}{\alpha_n} L_n(f_1) - L_n\left(\frac{f_1}{\alpha_n}\right) \right) \right| &\le \sum_{k=0}^{\infty} \alpha_n \|g_1\| \gamma_n L_n^k(\psi) \\ &\le \frac{\alpha_n}{\nu_n} \cdot \|g_1\| \gamma_n \psi \\ &\le (1+\eta_n)\gamma_n \|g_1\| \psi. \end{aligned}$$

We obtain (30). Write

$$\alpha_n G_n(h_n) = \alpha_n G_n(h_n - \psi g) + \alpha_n G_n(\psi g). \tag{31}$$

By Theorem 4,

$$\lim_{n\to\infty} \|\alpha_n G_n(\psi g) - 2F(g)\|_\psi = 0. \tag{32}$$

From Corollary 1 there is a constant M such that $\|\alpha_n G_n\|_{\mathcal{L}(C_\psi[0,1],C_\psi[0,1])} \leq M$, $n \in \mathbb{N}$. It follows $\|\alpha_n G_n(h_n - \psi g)\|_\psi \leq M\|h_n - \psi g\|_\psi$ and from relation (29) we obtain

$$\lim_{n\to\infty} \|\alpha_n G_n(h_n - \psi g)\|_\psi = 0. \tag{33}$$

Now from relations (30), (31), (32) and (33) we obtain

$$-f_1 = \lim_{n\to\infty} \alpha_n G_n(h_n) = 2F(g).$$

Since $2F(g) \in C^2(0,1)$, it follows $-f_1 \in C^2(0,1]$ and consequently $f \in C^2(0,1)$. Moreover, using Lemma 3 we deduce on interval $(0,1)$: $f'' = (f_1 + B_1(f))'' = (f_1)'' = (-2F(g))'' = 2g$. $\square$

6 Applications

6.1 Bernstein operators

The classical Bernstein operators are given by

$$B_n(f)(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) p_{n,k}(x), \ f \in C[0,1], \ x \in [0,1],$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $0 \leq k \leq n$.

If we denote the moments of operators B_n, by $M_n^k(x)$, $x \in [0,1]$, $k = 0,1,\ldots$, then $M_n^2(x) = \frac{\psi(x)}{n}$, $M_n^4(x) = \frac{\psi(x)}{n^2}\left[\left(3 - \frac{6}{n}\right)\psi(x) + \frac{1}{n}\right]$. It follows $B_n \in \Lambda_0$, with $\alpha_n(x) = \nu_n = \frac{1}{n}$. Then, conditions (18) and (28) are automatically satisfied. Also, for $n \geq 2$, we have $M_n^4(x)/M_n^2(x) = \left(\frac{3}{n} - \frac{6}{n^2}\right)\psi(x) + \frac{1}{n^2} \leq \frac{3}{4}\cdot\frac{1}{n} - \frac{1}{2}\cdot\frac{1}{n^2}$. Hence relation (17) is satisfied too. Then we can apply Theorem 4 and Theorem 5 to Bernstein operators.

6.2 Bernstein-Durrmeyer type operators

A class of Bernstein-Durrmeyer type operators U_n^ρ, $\rho \in (0,\infty)$, $n \in \mathbb{N}$, which preserve linear functions is considered in [25], [16]:

$$(U_n^\rho f)(x) = \sum_{k=1}^{n-1} F_{n,k}^\rho(f) p_{n,k}(x) + f(0)p_{n,0}(x) + f(1)p_{n,n}(x), \ f \in C[0,1], \ x \in [0,1]$$

with functionals $F_{n,k}^\rho$, $1 \leq k \leq n-1$, defined by:

$$F_{n,k}^\rho(f) := \int_0^1 f(t)\mu_{n,k}^\rho(t)dt, \ \mu_{n,k}^\rho(t) := \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)},$$

where $B(x,y)$ is Euler's Beta function.

We mention that in the special case $\rho = 1$, one obtains the "genuine" Durrmeyer operator and for each $f \in C[0,1]$ we have $\lim_{\rho\to\infty} U_n^\rho(f) = B_n(f)$, $f \in C[0,1]$, uniformly.

For fixed ρ, let denote the moments of operators U_n^ρ, by $M_n^k(x)$, $n \in \mathbb{N}$, $k = 0, 1, \ldots$, $x \in [0,1]$. In [25], [16] the following formulas are derived:

$$\begin{aligned}
&M_n^0(x) = 1, \\
&M_n^1(x) = 0, \\
&M_n^2(x) = \frac{(\rho+1)x(1-x)}{n\rho+1}, \\
&M_n^4(x) = \frac{3\rho(\rho+1)^2\psi^2(x)n}{(n\rho+1)(n\rho+2)(n\rho+3)} + \\
&+\frac{-6(\rho+1)(\rho^2+3\rho+3)\psi^2(x)+(\rho+1)(\rho+2)(\rho+3)\psi(x)}{(n\rho+1)(n\rho+2)(n\rho+3)}.
\end{aligned}$$

Hence $U_n^\rho \in \Lambda_0$ with $\alpha_n = \nu_n = \frac{\rho+1}{n\rho+1}$. Then relation (18) and (28) are automatically satisfied. Condition (17) is also satisfied, since

$$M_n^4(x)/M_n^2(x) \le \frac{3\rho(\rho+1)n\psi(x) - 6(\rho^2+3\rho+3)\psi(x) + (\rho+2)(\rho+3)}{(n\rho+2)(n\rho+3)}$$

and the functions given in the right side of this inequality converge uniformly to 0. Consequently Theorem 4 and Theorem 5 can be applied to operators U_n^ρ.

6.3 A symmetrical version of the Meyer-König and Zeller operators

The Meyer-König and Zeller operators [20], with a slight modification made by Cheney and Sharma [9] are defined, for $f \in C[0,1]$, by

$$Z_n(f)(x) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} \binom{n+k}{k} (1-x)^{n+1} x^k f\left(\frac{k}{n+k}\right), & x \in [0,1), \\ f(1), & x = 1. \end{cases} \tag{34}$$

These operators reproduce linear functions.

We consider here a symmetric variant of operators Z_n. Let $\tau := e_0 - e_1$ and define

$$Z_n^1(f)(x) := Z_n(f \circ \tau)(1-x),\ f \in C[0,1], \quad x \in [0,1], \tag{35}$$

$$Z_n^\star(f) := \frac{1}{2}(Z_n + Z_n^1). \tag{36}$$

We have $Z_n^1(e_0) = Z_n(e_0) = e_0$ and, for $x \in [0,1]$: $Z_n^1(e_1)(x) = Z_n(e_0 - e_1)(1-x) = 1 - (1-x) = e_1(x)$. Then Z_n^1 and consequently also Z_n^* preserve linear functions.

Let $n \in \mathbb{N}$, $k \in \mathbb{N}$ and $x \in [0,1]$. Denote $m_n^k(x) = Z_n((e_1 - xe_0)^k)(x)$ and $M_n^k(x) = Z_n^\star((e_1 - xe_0)^k)(x)$. We have $Z_n^1((e_1 - xe_0)^k)(x) = (-1)^k m_n^k(1-x)$. Therefore, if k is even we have $M_n^k(x) = \frac{1}{2}(m_n^k(x) + m_n^k(1-x))$. Consider also the notations given in (14) and (15) for operators $L_n = Z_n^\star$, $n \in \mathbb{N}$.

Becker and Nessel [7] proved the double inequality:

$$\frac{x(1-x)^2}{n+1}\left(1 + \frac{2x}{n+2}\right) \le m_n^2(x) \le \frac{x(1-x)^2}{n+1}\left(1 + \frac{2x}{n+1}\right). \tag{37}$$

From this we deduce

$$\frac{\psi}{2(n+1)}\left(1+\frac{4\psi}{n+2}\right) \le M_n^2(x) \le \frac{\psi}{2(n+1)}\left(1+\frac{4\psi}{n+1}\right).$$

Hence $M_n^2(x)=\alpha_n\psi$, $\alpha_n \in B[0,1]\cap C(0,1)$ and $\frac{1}{2(n+1)} \le \alpha_n(x) \le \frac{n+2}{2(n+1)^2}$, for $x\in[0,1]$. It follows $Z_n^\star \in \Lambda$. Moreover, it follows that $\eta_n \le \frac{1}{n+1}$, hence, the sequence of operators $(Z_n^\star)_n$ satisfies condition (18).

We start now to estimate the values of $Z_n(e_r)$, for an arbitrary $r\ge 2$. A complete asymptotic expansion is given by Abel in [1]. Here we need a more simple estimate, but in which the factor ψ is taken into account. In particular case we obtain a new estimate of moment $m_n^2(x)$, which, from certain point of view is more precise then the estimate of Becker and Nessel (37).

Let $x^{\underline{j}} = x(x-1)\dots(x-j+1)$. Let $s(j,i)$ and $S(j,i)$, $1\le i\le j$ be the Stirling numbers of the first kind and of the second kind, respectively: $x^{\underline{j}} = \sum\limits_{i=1}^{j} s(j,i)x^i$ and $x^j=\sum_{i=1}^j S(j,i)x^{\underline{i}}$. Let $r\ge 2$, $n\ge 3$ and $x\in[0,1)$. We have

$$\begin{aligned}
Z_n(e_r)(x) &= \sum_{k=0}^{\infty}\binom{n+k}{k}(1-x)^{n+1}x^k\left(\frac{k}{n+k}\right)^r\\
&= \sum_{j=1}^{r} S(r,j)\sum_{k=0}^{\infty}\binom{n+k}{k}(1-x)^{n+1}x^k\frac{k^{\underline{j}}}{(n+k)^r}\\
&= \sum_{j=1}^{r} S(r,j)\sum_{k=j}^{\infty}\binom{n+k-j}{k-j}(1-x)^{n+1}x^k\frac{(n+k)^{\underline{j}}}{(n+k)^r}\\
&= \sum_{j=1}^{r} S(r,j)\sum_{i=1}^{j}s(j,i)\sum_{k=0}^{\infty}\binom{n+k}{k}(1-x)^{n+1}x^{k+j}\frac{(n+k+j)^i}{(n+k+j)^r}\\
&= \sum_{i=1}^{r}\sum_{j=i}^{r} S(r,j)s(j,i)\sum_{k=0}^{\infty}\binom{n+k}{k}(1-x)^{n+1}x^{k+j}\frac{1}{(n+k+j)^{r-i}}\\
&=: \sum_{i=1}^{r} T_i.
\end{aligned}$$

The permutation of the sums is possible due to the absolute convergence of the involved series. Then

$$T_r = S(r,r)s(r,r)\sum_{k=0}^{\infty}\binom{n+k}{k}(1-x)^{n+1}x^{k+r} = x^r.$$

In order to estimate term T_{r-1} we use the following formula

$$\binom{n+k}{k}\frac{1}{n+k+j} = \binom{n+k-1}{k}\frac{1}{n} + \binom{n+k-2}{k}\left[-\frac{j}{n(n-1)}+\frac{j(j+1)}{n(n-1)(n+k+j)}\right].$$

Denote

$$\sigma_{n,r}(x) = \sum_{j=r-1}^{r} S(r,j)s(j,r-1)\sum_{k=0}^{\infty}\binom{n+k-2}{k}(1-x)^{n+1}x^{k+j}\frac{j(j+1)}{n(n-1)(n+k+j)}.$$

We obtain

$$\begin{aligned}
T_{r-1} &= \sum_{j=r-1}^{r} S(r,j)s(j,r-1)\left[\frac{1}{n}\cdot x^j(1-x) - \frac{j}{n(n-1)}\cdot x^j(1-x)^2\right] + \sigma_{n,r}(x) \\
&= \binom{r}{2}\sum_{j=r-1}^{r}(-1)^{r-j-1}x^j(1-x)\left[\frac{1}{n} - \frac{j}{n(n-1)}\cdot(1-x)\right] + \sigma_{n,r}(x) \\
&= \binom{r}{2}x^{r-1}(1-x)^2\left[\frac{1}{n} - \frac{1}{n(n-1)}(r-1-rx)\right] + \sigma_{n,r}(x).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
|\sigma_{n,r}(x)| &\leq \sum_{j=r-1}^{r}\binom{r}{2}\sum_{k=0}^{\infty}\binom{n+k-2}{k}(1-x)^{n+1}\frac{r(r+1)x^{k+r-1}}{n(n-1)(n+r-1)} \\
&= 2\binom{r}{2}\frac{r(r+1)}{n(n-1)(n+r-1)}\cdot x^{r-1}(1-x)^2 \\
&\leq \frac{r^2(r^2-1)}{n(n^2-1)}\cdot x^{r-1}(1-x)^2.
\end{aligned}$$

Finally we have

$$\begin{aligned}
\left|\sum_{i=1}^{r-2}T_i\right| &= \left|\sum_{i=1}^{r-2}\sum_{j=i}^{r}S(r,j)s(j,i)\sum_{k=0}^{\infty}\binom{n+k}{k}(1-x)^{n+1}x^{k+j}\cdot\frac{1}{(n+k+j)^{r-i}}\right| \\
&= \left|\sum_{i=1}^{r-2}\sum_{j=i}^{r}S(r,j)s(j,i)\frac{1}{n(n-1)}\sum_{k=0}^{\infty}\binom{n+k-2}{k}(1-x)^{n+1}x^{k+j}\frac{(n+k)^{\underline{2}}}{(n+k+j)^{r-i}}\right| \\
&\leq \frac{1}{n(n-1)}\sum_{i=1}^{r-2}\sum_{j=i}^{r}|S(r,j)s(j,i)|x^j(1-x)^2 \\
&\leq \frac{x(1-x)}{n(n-1)}\sum_{i=1}^{r-2}\sum_{j=i}^{r}|S(r,j)s(j,i)|.
\end{aligned}$$

From relations above we obtain:

$$Z_n(e_r)(x) = x^r + \frac{1}{n}\binom{r}{2}x^{r-1}(1-x)^2 + \varepsilon_{n,r}(x) \text{ with } |\varepsilon_{n,r}(x)| \leq \frac{x(1-x)}{n(n-1)}\cdot C_r, \tag{38}$$

where C_r is a constant depending only on r. This is obviously true also for $x=1$.

In the case $r = 2$ we obtain an estimate more refined than (38), which leads to:

$$m_n^2(x) = x(1-x)^2 \left[\frac{1}{n} - \frac{1-2x}{n(n-1)} + \rho_n(x) \right] \text{ with } |\rho_n(x)| \le \frac{12}{n(n^2-1)}. \tag{39}$$

Denote $\mu_n(x) = \frac{1}{2}((1-x)\rho_n(x) + x\rho_n(1-x))$. We obtain $M_n^2(x) = \psi(x)\alpha_n(x)$, where

$$\alpha_n(x) = \frac{1}{2n} - \frac{(1-2x)^2}{2n(n-1)} + \mu_n(x) \ \text{ with } \ |\mu_n(x)| \le \frac{6}{n(n^2-1)}.$$

Also, for $t, x \in [0,1]$, we obtain

$$\begin{aligned} |\alpha_n(t) - \alpha_n(x)| &\le \frac{2}{n(n-1)} \cdot |(t-x)(t+x-1)| + |\mu_n(t)| + |\mu_n(x)| \\ &\le \frac{2|t-x|}{n(n-1)} + \frac{12}{n(n^2-1)}. \end{aligned}$$

Note that $Z_n^\star(\psi(e_1 - xe_0)^2)(x) \le \frac{1}{4} Z_n^\star((e_1 - xe_0)^2)(x) \le \frac{1}{4} \cdot \psi(x)\|\alpha_n\|$ and $Z_n^\star(\psi)(x) \le \psi(x)$. From the relation above we obtain

$$\begin{aligned} Z_n^\star(\psi|\alpha_n - \alpha_n(x)e_0|)(x) &\le \frac{2}{n(n-1)} \cdot Z_n^\star(\psi|e_1 - xe_0|)(x) + \frac{12}{n(n^2-1)} \cdot Z_n^\star(\psi)(x) \\ &\le \frac{2}{n(n-1)} \sqrt{Z_n^\star(\psi)(x) Z_n^\star(\psi(e_1 - xe_0)^2)(x)} + \frac{12\psi(x)}{n(n^2-1)} \\ &\le \frac{\psi(x)}{n(n-1)} \cdot \left[\sqrt{\|\alpha_n\|} + \frac{12}{n+1} \right]. \end{aligned}$$

Then relation (28) is immediate.

From relation (38), for $r = 2, 3, 4$, we obtain, that there is a bounded function $q_n(x)$ with $|q_n(x)| \le M$, $x \in [0,1]$, such that

$$m_n^4(x) = \frac{x(1-x)}{n(n-1)} \cdot q_n(x).$$

Since $M_n^4(x) = \frac{x(1-x)}{2n(n-1)} \cdot (q_n(x) + q_n(1-x))$, it follows $M_n^4(x) \le \frac{M}{n(n-1)\nu_n} \cdot M_n^2(x)$. Therefore, relation (17) is valid.

Consequently, all the conditions in Theorem 4 and Theorem 5 are satisfied for the sequence of operators $(Z_n^\star)_{n\ge 3}$.

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Ulrich Abel
Technische Hochschule Mittelhessen, University of Applied Sciences,
Department MND, Wilhelm-Leuschner-Straße 13,
61169 Friedberg, Germany
e-mail: Ulrich.Abel@mnd.thm.de

Mircea Ivan
Technical University of Cluj-Napoca,
Department of Mathematics, Str. Memorandumului nr. 28,
400114 Cluj-Napoca, Romania
e-mail: Mircea.Ivan@math.utcluj.ro

Radu Păltănea
"Transilvania" University of Braşov

Faculty of Mathematics and Computer Science,
Str. Iuliu Maniu nr. 50,
500091 Braşov, Romania
e-mail: radupaltanea@yahoo.com